

THE MOTIVIC ADAMS VANISHING LINE OF SLOPE $1/2$

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ABSTRACT. We establish a motivic version of Adams' vanishing line of slope $1/2$ in the cohomology of the motivic Steenrod algebra over \mathbb{C} .

1. INTRODUCTION

One of the most powerful tools for computing stable homotopy groups of spheres is the Adams spectral sequence

$$\mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \hat{\pi}_{t-s},$$

where A denotes the Steenrod algebra of stable mod 2 cohomology operations and $\hat{\pi}_*$ are the 2-completed stable homotopy groups. Adams established that these Ext groups vanish above a certain line of slope $1/2$, with the exception of the elements h_0^k in the 0-stem [A2].

In the motivic context over \mathbb{C} , inspection of an Adams chart [I2] immediately shows that the analogous Ext groups do not vanish above the same line of slope $1/2$. (However, the motivic Ext groups do vanish above a line of slope 1, analogously to an earlier classical vanishing result of Adams [A1].) Further inspection shows that in a large range, all elements above the Adams line of slope $1/2$ are h_1 -local, in the sense that they are h_1 -divisible and support infinitely many multiplications by h_1 . This suggests the following theorem, whose proof is the goal of this article.

Theorem 1.1. *Let $s > 0$, and let A be the motivic mod 2 Steenrod algebra over \mathbb{C} . The map*

$$h_1 : \mathrm{Ext}_A^{s,f,w}(\mathbb{M}_2, \mathbb{M}_2) \rightarrow \mathrm{Ext}_A^{s+1,f+1,w+1}(\mathbb{M}_2, \mathbb{M}_2)$$

is an isomorphism if $f \geq \frac{1}{2}s + 2$, and it is a surjection if $f \geq \frac{1}{2}s + \frac{1}{2}$.

In the grading (s, f, w) from Theorem 1.1, s is the stem and f is the Adams filtration so that (s, f) represent the traditional coordinates in an Adams chart. Meanwhile, w is the motivic weight, which is not relevant to the statement of the theorem.

Proof. In Section 7, we will show that

$$\mathrm{Ext}_A^{s,f,w}(\mathbb{M}_2, \mathbb{M}_2) \cong \mathrm{Ext}_A^{s-1,f-1,w-1}(N, \mathbb{M}_2),$$

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where N is a certain A -module that is free as a left $A(0)$ -module. We will prove in Proposition 7.2 that multiplication by h_1 has the desired property for such A -modules. \blacksquare

Inspection shows that Theorem 1.1 is optimal in the following sense. Multiplication by h_1 is not an isomorphism along the line $f = \frac{1}{2}s + \frac{3}{2}$, and it is not a surjection along the line $f = \frac{1}{2}s$.

This article is only concerned with algebraic calculations of Ext groups and does not discuss Adams spectral sequences for which these Ext groups are inputs. However, our interest in h_1 multiplications is motivated by recent work [AM2] [GI] on the η -local motivic sphere, where η is the first Hopf map that is detected by h_1 .

Another possible approach to Theorem 1.1 uses the methods established by Andrews and Miller [AM2] (see especially [AM2, Lemma 5.2.1]). However, one must be careful about the slightly strange behavior of motivic Margolis homology (see Proposition 3.2). For the techniques of [AM2] to work motivically, one needs to verify that certain quotients of the motivic Steenrod algebra are \mathbb{M}_2 -free. This is more or less the same as the homological results of Sections 3 and 4.

1.1. Organization. We review some background in Section 2. In Sections 3 through 6, we assemble various facts about motivic homological algebra. Finally, we give the main technical result in Proposition 7.2, from which Theorem 1.1 follows immediately.

2. PRELIMINARIES

2.1. Notation. We continue with notation from [I1] as follows:

- (1) \mathbb{M}_2 is the motivic cohomology of \mathbb{C} with \mathbb{F}_2 coefficients.
- (2) A is the mod 2 motivic Steenrod algebra over \mathbb{C} generated by elements Sq^1 of bidegree $(1, 0)$ and Sq^{2^n} of bidegree $(2^n, 2^{n-1})$ for $n \geq 1$.
- (3) $A(0)$ is the (exterior) \mathbb{M}_2 -subalgebra of A generated by Sq^1 .
- (4) $A(1)$ is the \mathbb{M}_2 -subalgebra of A generated by Sq^1 and Sq^2 .
- (5) Ext_A is the trigraded ring $\text{Ext}_A(\mathbb{M}_2, \mathbb{M}_2)$.

The following two theorems of Voevodsky are the starting points of our calculations.

Theorem 2.1 ([V1]). \mathbb{M}_2 is the bigraded ring $\mathbb{F}_2[\tau]$, where τ has bidegree $(0, 1)$.

It is often more convenient to work with the dual $A_{*,*} = \text{Hom}_{\mathbb{M}_2}(A, \mathbb{M}_2)$, which was described by Voevodsky. (See also [B] for a clean description.)

Theorem 2.2. [V2] [V3, Theorem 12.6] *The dual motivic Steenrod algebra $A_{*,*}$ is generated as an \mathbb{M}_2 -algebra by $\xi_i \in A_{2(2^i-1), 2^i-1}$ and $\tau_i \in A_{2^{i+1}-1, 2^i-1}$ subject to the relations*

$$\tau_i^2 = \tau \xi_{i+1}.$$

The coproduct is given on the generators by the following formulas, in which $\xi_0 = 1$:

$$\Delta(\tau_k) = \tau_k \otimes 1 + \sum_i \xi_{k-i}^{2^i} \otimes \tau_i$$

$$\Delta(\xi_k) = \sum_i \xi_{k-i}^{2^i} \otimes \xi_i.$$

Remark 2.3. The quotient $A_{*,*}/\tau = A_{*,*} \otimes_{\mathbb{M}_2} \mathbb{F}_2$ is analogous to the odd-primary classical dual Steenrod algebra, in the sense that there is an infinite family of exterior generators τ_i and an infinite family of polynomial generators ξ_i . On the other hand, the localization $A_{*,*}[\tau^{-1}]$ is analogous to the mod 2 classical dual Steenrod algebra, which has only polynomial generators τ_i .

2.2. Grading conventions. We follow [I1] in grading Ext_A according to (s, f, w) , where:

- (1) f is the Adams filtration, i.e., the homological degree.
- (2) $s + f$ is the internal degree, i.e., corresponds to the first coordinate in the bidegrees of A .
- (3) s is the stem, i.e., the internal degree minus the Adams filtration.
- (4) w is the weight.

3. MARGOLIS HOMOLOGY

Recall that $\text{Sq}^1 \text{Sq}^1 = 0$, so that Sq^1 acts as a differential on any A -module. We write $H^{**}(M; \text{Sq}^1)$ for the resulting **Margolis homology** groups [AM1]. We say that an A -module is **bounded below** if $M^{p,q} = 0$ for sufficiently small p . A bounded below A -module is **of finite type** if each $M^{p,*}$ is a finitely generated \mathbb{M}_2 -module.

Remark 3.1. We will need the following fact about finitely generated \mathbb{M}_2 -modules. Such modules are of the form

$$(\mathbb{M}_2)^k \oplus \bigoplus_i (\mathbb{M}_2 / \tau^{r_i}).$$

This is a graded version of the classification of finitely generated modules over a principal ideal domain, since every homogeneous ideal of \mathbb{M}_2 is generated by an element of the form τ^{r_i} . Consequently, a finitely generated \mathbb{M}_2 -module is free if and only if it has no τ torsion.

Proposition 3.2. *Let M be a bounded below A -module of finite type. Then M is free as an $A(0)$ -module if and only if M is free as an \mathbb{M}_2 -module and $H^{*,*}(M; \text{Sq}^1) = 0$.*

Proof. The forward implication is clear. Thus suppose that M is free over \mathbb{M}_2 and $H^{*,*}(M; \text{Sq}^1) = 0$. Suppose that M is concentrated in degrees (p, q) with $p \geq n_0$. Let x be a nonzero element of $M^{n_0,*}$ of smallest weight, and let $y = \text{Sq}^1(x)$ in $M^{n_0+1,*}$. The map $\text{Sq}^1 : M^{n_0,*} \rightarrow M^{n_0+1,*}$ is injective because $H^{*,*}(M; \text{Sq}^1) = 0$, so y is non-zero.

Let N be the \mathbb{M}_2 -submodule of M generated by x and y , and let P be the quotient M/N . Then N is a free $A(0)$ -module generated by x .

We will next argue that P is \mathbb{M}_2 -free; by Remark 3.1, this is the same as showing that P has no τ torsion. Equivalently, we will show that if τz belongs to N , then so does z . The main point is that y is not divisible by τ . Suppose for sake of contradiction that $\tau z = y$. Then $\tau \text{Sq}^1(z) = \text{Sq}^1(y) = 0$. Since M is \mathbb{M}_2 -free, Remark 3.1 implies that $\text{Sq}^1(z) = 0$. On the other hand, z cannot be in the image of Sq^1 because its weight is less than the weight of x . This contradicts the assumption that $H^{*,*}(M; \text{Sq}^1)$ is zero.

This establishes that M is isomorphic to $N \oplus P$ as an $A(0)$ -module. The long exact sequence associated to the short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$$

shows that $H^{*,*}(P; \text{Sq}^1) = 0$.

Having split M as $N \oplus P$, we may now apply the same argument to P to split off another free $A(0)$ -module. The finite type assumption on M guarantees that this process eventually splits all of M into free $A(0)$ -modules. \blacksquare

Corollary 3.3. *Suppose that*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is a short exact sequence of bounded below A -modules of finite type which are free over \mathbb{M}_2 . If any two modules in this sequence are free over $A(0)$, then so is the third.

Proof. Two of the modules have no Margolis homology by Proposition 3.2. The long exact sequence in Margolis homology shows that the third also has no Margolis homology. Then Proposition 3.2 again implies that the third module is free over $A(0)$. \blacksquare

4. THE MOTIVIC MILNOR BASIS

Let $E = (\epsilon_0, \epsilon_1, \epsilon_2, \dots)$ be a sequence of ones and zeros, almost all zero, and let $R = (r_1, r_2, \dots)$ be a sequence of nonnegative integers, almost all zero. Then, according to Theorem 2.2, the elements

$$\tau(E)\xi(R) := \prod_{i \geq 0} \tau_i^{\epsilon_i} \prod_{j \geq 1} \xi_j^{r_j}$$

give an \mathbb{M}_2 -basis for $A_{*,*}$. We follow [DI] in writing $P^{(\epsilon_0+2r_1, \epsilon_1+2r_2, \dots)}$ for the corresponding elements of the dual \mathbb{M}_2 -basis for A . These elements form the **motivic Milnor basis** for A . The resulting \mathbb{M}_2 -basis for $A(1)$ consists of the elements $P^{(s_1, s_2, 0, 0, \dots)}$ such that $0 \leq s_1 \leq 3$ and $0 \leq s_2 \leq 1$.

Let $S = (s_1, s_2, \dots)$. The **excess** of the Milnor basis element P^S is defined to be $e(P^S) = \sum_i s_i$. We extend this to arbitrary elements of A by taking the excess of an element θ to be the maximal excess of any Milnor basis element appearing in an expression for θ .

For two sequences $R = (r_1, r_2, \dots)$ and $S = (s_1, s_2, \dots)$, we write $R + S$ for the termwise sum $(r_1 + s_1, r_2 + s_2, \dots)$.

Lemma 4.1. *Let $R = (r_1, r_2, \dots)$ and $S = (s_1, s_2, \dots)$. Then*

$$P^R \cdot P^S = \prod_i \binom{r_i}{s_i} \cdot P^{R+S} + \text{terms of lower excess.}$$

Proof. This follows from the description of the dual motivic Steenrod algebra, similarly to the classical case [M]. \blacksquare

We will use the Milnor basis to establish the following fact about the right action of $A(1)$ on A .

Proposition 4.2. *The motivic Steenrod algebra A is free as a right $A(1)$ -module.*

Proof. Define a filtration on A by $F_s(A) = \{\theta \in A \mid e(\theta) \leq s\}$. Lemma 4.1 implies that the associated graded object

$$\mathrm{gr}_* A = \bigoplus_s F_s(A)/F_{s-1}(A)$$

inherits the structure of an \mathbb{M}_2 -algebra. Let M be the \mathbb{M}_2 -submodule of A generated by Milnor basis elements of the form $P^{(4s_1, 2s_2, s_3, s_4, \dots)}$. Consider the composition

$$M \otimes_{\mathbb{M}_2} A(1) \longrightarrow A \longrightarrow \mathrm{gr}_* A,$$

in which the first map is induced by the multiplication of A and the second map is an isomorphism of \mathbb{M}_2 -modules.

Lemma 4.1 implies that the composition is an isomorphism of \mathbb{M}_2 -modules. Therefore, the first map is also an isomorphism. This shows that A is free as a right $A(1)$ -module with basis consisting of elements of the form $P^{(4s_1, 2s_2, s_3, s_4, \dots)}$. ■

Remark 4.3. The reader may wonder why Proposition 4.2 is not an immediate consequence of results in [MM]. The problem is that it is not obvious that the projection $A \rightarrow A//A(1)$ is split as an \mathbb{M}_2 -map. The proof of Proposition 4.2 is essentially the same as showing that the projection is in fact split.

Example 4.4. Let B be the Hopf subalgebra of A that is generated over \mathbb{M}_2 by $\tau \mathrm{Sq}^1$. The projection $A \rightarrow A//B$ is not split as an \mathbb{M}_2 -map, since Sq^1 projects to an element that is τ torsion. Moreover, A is not free as a right B -module.

5. THE MODULE \tilde{A} .

Definition 5.1. Let $\tilde{A}(1)$ be the left $A(1)$ -module on two generators a and b of degrees $(0, 0)$ and $(2, 0)$ respectively, subject to the relations

$$\mathrm{Sq}^2 a = \tau b, \quad \mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^1 a = \mathrm{Sq}^2 b.$$

The relation $\mathrm{Sq}^2 a = \tau b$ implies that $\mathrm{Sq}^2 \mathrm{Sq}^2 a = \tau \mathrm{Sq}^2 b$, so $\tau \mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^1 a = \tau \mathrm{Sq}^2 b$. However, the first relation does not imply that $\mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^1 a = \mathrm{Sq}^2 b$. This explains why we need a second relation in the definition of $\tilde{A}(1)$.

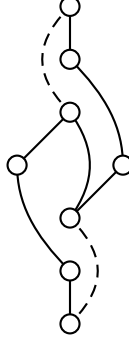
Figure 1 represents $\tilde{A}(1)$, according to the following key:

- (1) Each circle represents a copy of \mathbb{M}_2 .
- (2) Each straight line represents multiplication by Sq^1 .
- (3) Each curved line represents multiplication by Sq^2 .
- (4) Each dashed line indicates that the squaring operation hits τ times an \mathbb{M}_2 -generator, but not the generator itself.

Analogous to the Milnor basis for $A(1)$, we have a basis for $\tilde{A}(1)$ consisting of the elements $\tilde{P}^{(s_1, s_2)}$ such that $0 \leq s_1 \leq 3$ and $0 \leq s_2 \leq 1$. The elements \tilde{P}^2 , \tilde{P}^3 , $\tilde{P}^{2,1}$, and $\tilde{P}^{3,1}$ have weight one less than the corresponding Milnor basis elements for $A(1)$. We define the excess in $\tilde{A}(1)$ using this basis.

Remark 5.2. Just like in the classical case [DM], the $A(1)$ -modules $A(1)$ and $\tilde{A}(1)$ each extend to A -modules in four different ways, determined by the action of Sq^4 . Adams spectral sequence computations verify that all eight of these A -modules arise as the cohomology of a 2-complete motivic spectrum. These constructions are the subject of work in progress on motivic v_1 -self maps.

Definition 5.3. Let \tilde{A} be the left A -module $A \otimes_{A(1)} \tilde{A}(1)$.

FIGURE 1. $\tilde{A}(1)$

Remark 5.4. Proposition 4.2 implies that \tilde{A} is the left A -module generated by two elements a and b of degrees $(0, 0)$ and $(2, 0)$ subject to the relations

$$\mathrm{Sq}^2 a = \tau b, \quad \mathrm{Sq}^3 \mathrm{Sq}^1 a = \mathrm{Sq}^2 b.$$

Proposition 5.5. *The A -module \tilde{A} is free as a left $A(0)$ -module.*

Proof. We have a Milnor-style basis for \tilde{A} consisting of elements of the form $P^{(4r_1, 2r_2, r_3, \dots)} \otimes \tilde{P}^{s_1, s_2}$ such that $0 \leq s_1 \leq 3$ and $0 \leq s_2 \leq 1$. We also have a filtration by excess.

Lemma 4.1 implies that

$$P^1 P^{(4r_1, 2r_2, r_3, \dots)} \equiv P^{(4r_1+1, 2r_2, r_3, \dots)} \equiv P^{(4r_1, 2r_2, r_3, \dots)} P^1$$

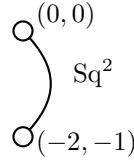
in A , modulo terms of lower excess. Also, $P^1 \tilde{P}^{s_1, s_2} = \tilde{P}^{s_1+1, s_2}$ in $\tilde{A}(1)$ if s_1 is even. Therefore, if s_1 is even, then

$$P^1 (P^{(4r_1, 2r_2, r_3, \dots)} \otimes \tilde{P}^{s_1, s_2}) \equiv P^{(4r_1, 2r_2, r_3, \dots)} \otimes \tilde{P}^{s_1+1, s_2}$$

in \tilde{A} , modulo terms of lower excess. It follows that an $A(0)$ -basis for \tilde{A} consists of elements of the form $P^{(4r_1, 2r_2, r_3, \dots)} \otimes \tilde{P}^{s_1, s_2}$ such that s_1 is even. \blacksquare

6. THE COFIBER OF η

Definition 6.1. Let $C(\eta)$ denote the cofiber of the first Hopf map $\eta : S^{1,1} \rightarrow S^{0,0}$. We then write \mathbf{C}_η for the A -module $\Sigma^{-2, -1} \mathrm{H}^{*,*}(C(\eta))$. Thus \mathbf{C}_η has a bottom cell in bidegrees $(-2, -1)$ and a top cell in $(0, 0)$, connected by a Sq^2 (see figure).



The following result implies that, for any A -module M , the groups $\mathrm{Ext}^*(M, \mathbf{C}_\eta)$ may be computed using a resolution $F^* \rightarrow M$ whose terms are of the form $F^n \cong A^r \oplus \tilde{A}^s$.

Proposition 6.2. $\mathrm{Ext}_A^{s,f}(\tilde{A}, \mathbf{C}_\eta) = 0$ for $f > 0$.

Proof. The kernel of the defining quotient $A(1) \oplus \Sigma^{2,0}A(1) \longrightarrow \tilde{A}(1)$ is isomorphic to $\Sigma^{2,1}\tilde{A}(1)$. It follows that we can define a periodic free $A(1)$ -resolution

$$\tilde{A}(1) \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots,$$

where $F_n \cong A(1)\{x_n, y_n\}$, with $|x_n| = (2n, n)$, $|y_n| = (2n+2, n)$,

$$d(x_n) = \text{Sq}^2(x_{n-1}) + \tau \cdot y_{n-1}, \quad \text{and} \quad d(y_n) = \text{Sq}^3\text{Sq}^1(x_{n-1}) + \text{Sq}^2(y_{n-1}).$$

Using this resolution, it is simple to verify that

$$\text{Ext}_{A(1)}^{s,f}(\tilde{A}(1), \mathbf{C}_\eta) = 0, \quad \text{for } f > 0.$$

Since A is free as a right $A(1)$ -module by Proposition 4.2, we get that

$$\text{Ext}_A^{s,f}(\tilde{A}, \mathbf{C}_\eta) \cong \text{Ext}_{A(1)}^{s,f}(\tilde{A}(1), \mathbf{C}_\eta) = 0 \quad \text{for } f > 0. \quad \blacksquare$$

The next lemma translates properties of h_1 multiplication into homological properties of the A -module \mathbf{C}_η .

Lemma 6.3. *Let $n \geq 0$ and let M be an A -module. The following are equivalent:*

(1) *The map*

$$h_1 : \text{Ext}_A^{s,f,w}(M, \mathbb{M}_2) \rightarrow \text{Ext}_A^{s+1,f+1,w+1}(M, \mathbb{M}_2)$$

is injective when $s < 2f - 2$ and $f \leq n$, and it is surjective when $s \leq 2f$ and $f < n$.

(2) $\text{Ext}_A^{s,f,w}(M, \mathbf{C}_\eta) = 0$ *when $s < 2f$ and $f \leq n$.*

Proof. The A -module \mathbf{C}_η sits in a short exact sequence

$$0 \longrightarrow \mathbb{M}_2 \longrightarrow \mathbf{C}_\eta \longrightarrow \Sigma^{-2,-1}\mathbb{M}_2 \longrightarrow 0.$$

This gives a long exact sequence

$$\xrightarrow{h_1} \text{Ext}_A^{s+1,f-1,w}(M, \mathbb{M}_2) \rightarrow \text{Ext}_A^{s+1,f-1,w}(M, \mathbf{C}_\eta) \rightarrow \text{Ext}_A^{s-1,f-1,w-1}(M, \mathbb{M}_2) \xrightarrow{h_1}$$

in which the connecting homomorphism is multiplication by h_1 . The result then follows easily. \blacksquare

Our next goal is to carry out an explicit low-dimensional Ext computation. This result will be critical for a later argument. We will consider $A(0)$ to be an A -module with the obvious action by A , i.e., Sq^1 acts by multiplication in the subalgebra $A(0)$, and Sq^n acts trivially for $n \geq 2$.

Proposition 6.4. $\text{Ext}_A^{s,f,w}(A(0), \mathbf{C}_\eta) = 0$ *when $s < 2f$ and $f \leq 4$.*

Proof. The short exact sequence

$$0 \longrightarrow \Sigma^{1,0}\mathbb{M}_2 \longrightarrow A(0) \longrightarrow \mathbb{M}_2 \longrightarrow 0$$

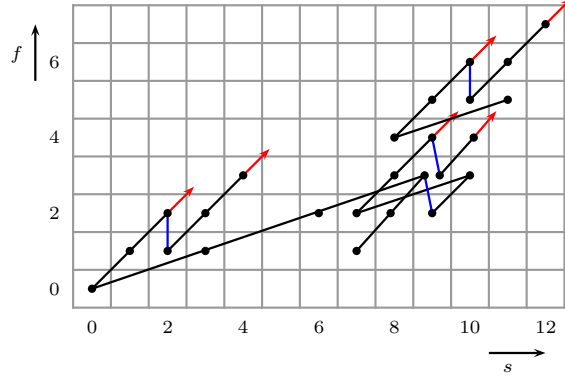
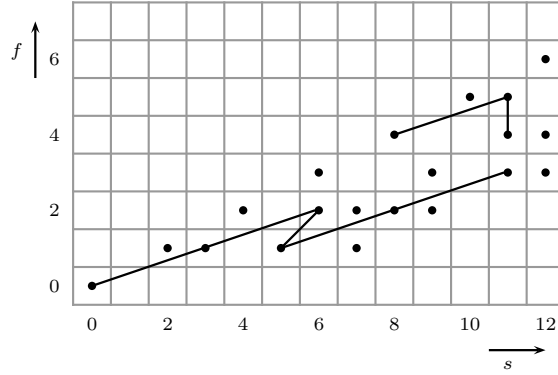
gives a calculation of $\text{Ext}(A(0), \mathbb{M}_2)$ in low degrees, starting from knowledge of $\text{Ext}(\mathbb{M}_2, \mathbb{M}_2)$ in a similar range. The short exact sequence

$$0 \longrightarrow \mathbb{M}_2 \longrightarrow \mathbf{C}_\eta \longrightarrow \Sigma^{-2,-1}\mathbb{M}_2 \longrightarrow 0$$

then yields a calculation of $\text{Ext}(A(0), \mathbf{C}_\eta)$. The results of these calculations are shown in Figures 2 and 3. \blacksquare

Figures 2 and 3 represent low-dimensional Ext calculations necessary for the proof of Proposition 6.4. Here is a key for reading the charts.

- (1) Black dots indicate copies of \mathbb{M}_2 .
- (2) Vertical lines indicate multiplications by h_0 .
- (3) Lines of slope 1 indicate multiplications by h_1 .
- (4) Lines of slope 1/3 indicate multiplications by h_2 .
- (5) Blue lines indicate that the multiplication hits τ times a generator.
- (6) Red arrows indicate infinitely many copies of \mathbb{M}_2/τ connected by h_1 multiplications.

FIGURE 2. $\text{Ext}_A(A(0), \mathbb{M}_2)$ FIGURE 3. $\text{Ext}_A(A(0), \mathbb{C}_\eta)$

7. MULTIPLICATION BY h_1 FOR $A(0)$ -FREE A -MODULES

Recall that $A//A(0)$ denotes the Hopf algebra quotient $A \otimes_{A(0)} \mathbb{M}_2$. There is a short exact sequence

$$0 \longleftarrow \mathbb{M}_2 \xleftarrow{\varepsilon} A//A(0) \longleftarrow I \longleftarrow 0$$

of A -modules, where ε is the augmentation and I is the augmentation ideal. This short exact sequence gives rise to a long exact sequence

$$\text{Ext}_A^{s,f,w} \xrightarrow{\varepsilon^*} \text{Ext}_A^{s,f,w}(A//A(0), \mathbb{M}_2) \longrightarrow \text{Ext}_A^{s,f,w}(I, \mathbb{M}_2) \xrightarrow{\partial} \text{Ext}_A^{s-1,f+1,w}$$

of modules over Ext_A . We have a change-of-rings isomorphism

$$\text{Ext}_A(A//A(0), \mathbb{M}_2) \cong \text{Ext}_{A(0)}(\mathbb{M}_2, \mathbb{M}_2) \cong \mathbb{M}_2[h_0],$$

so ε^* is an isomorphism when $s = 0$ and ∂ an isomorphism when $s > 1$.

The bottom class of I is Sq^2 , which occurs in bidegree $(2, 1)$, so that we may write $I \cong \Sigma^{2,1}N$ for a connective A -module N . It follows that for $s > 0$, we have a commutative square

$$\begin{array}{ccc}
 \text{Ext}_A^{s,f,w} & \xrightarrow{h_1} & \text{Ext}_A^{s+1,f+1,w+1} \\
 \cong \uparrow \partial & & \uparrow \partial \cong \\
 \text{Ext}_A^{s+1,f-1,w}(I, \mathbb{M}_2) & \xrightarrow{h_1} & \text{Ext}_A^{s+2,f,w+1}(I, \mathbb{M}_2) \\
 \cong \uparrow & & \uparrow \cong \\
 \text{Ext}_A^{s-1,f-1,w-1}(N, \mathbb{M}_2) & \xrightarrow{h_1} & \text{Ext}_A^{s,f,w}(N, \mathbb{M}_2)
 \end{array}$$

of Ext_A -module maps.

Lemma 7.1. *The A -module N is free as a left $A(0)$ -module.*

Proof. This argument is identical to the classical case.

We use the admissible \mathbb{M}_2 -basis for A , which consists of monomials of the form $\text{Sq}^{r_1}\text{Sq}^{r_2}\cdots\text{Sq}^{r_n}$ such that $r_i \geq 2r_{i+1}$. Then $A//A(0)$ has an \mathbb{M}_2 -basis consisting of the admissible monomials $\text{Sq}^{r_1}\text{Sq}^{r_2}\cdots\text{Sq}^{r_n}$ such that $r_n > 1$.

When r_1 is even, $\text{Sq}^1 \cdot \text{Sq}^{r_1}\text{Sq}^{r_2}\cdots\text{Sq}^{r_n}$ equals $\text{Sq}^{r_1+1}\text{Sq}^{r_2}\cdots\text{Sq}^{r_n}$. Therefore, the augmentation ideal I of $A//A(0)$ is a free left $A(0)$ -module with basis consisting of admissible monomials of the form $\text{Sq}^{r_1}\text{Sq}^{r_2}\cdots\text{Sq}^{r_n}$ such that r_1 is even and $r_n > 1$. \blacksquare

Proposition 7.2. *Let M be an A -module that is free as an $A(0)$ -module and concentrated in nonnegative degrees. Then the map*

$$h_1 : \text{Ext}_A^{s,f,w}(M, \mathbb{M}_2) \rightarrow \text{Ext}_A^{s+1,f+1,w+1}(M, \mathbb{M}_2)$$

is an isomorphism if $s < 2f - 2$, and it is a surjection if $s \leq 2f$.

We will mimic the classical argument of Adams [A2], with some variations to account for motivic phenomena.

Proof. By Lemma 6.3, it suffices to show that $\text{Ext}_A^{s,f,w}(M, \mathbf{C}_\eta)$ vanishes when $s < 2f$.

We begin by recalling from Proposition 6.4 that $\text{Ext}_A^{s,f,w}(A(0), \mathbf{C}_\eta)$ vanishes when $s < 2f$ and $f \leq 4$.

An arbitrary module M can be built up iteratively as an extension

$$0 \longrightarrow \bigoplus \Sigma^{t_i, w_i} A(0) \longrightarrow M \longrightarrow M' \longrightarrow 0$$

of A -modules, where $\text{Ext}_A^{s,f,w}(M', \mathbf{C}_\eta)$ vanishes for $s < 2f$ and $f \leq 4$ by induction. The long exact sequence in Ext then shows that $\text{Ext}_A^{s,f,w}(M, \mathbf{C}_\eta)$ also vanishes for $s < 2f$ and $f \leq 4$.

We have now established the proposition for $f \leq 4$. The next step is to extend the result to larger values of f . As in the previous step, we start with the special case $M = A(0)$.

In order to compute $\text{Ext}_A^{s,f,w}(A(0), \mathbf{C}_\eta)$, we must construct a resolution for $A(0)$. Proposition 6.2 says that this resolution can be built from copies of A or from copies of \tilde{A} .

We construct a resolution

$$A(0) \leftarrow R_0 \leftarrow R_1 \leftarrow R_2 \leftarrow R_3 \leftarrow \cdots$$

of $A(0)$ in the usual way by adding a copy of A to R_{n+1} for each indecomposable in the kernel K_n of the boundary map $R_n \rightarrow R_{n-1}$. However, when we find two indecomposable elements x and y of K_n such that $\text{Sq}^2 x = \tau y$, we add one copy of \tilde{A} to R_{n+1} rather than two copies of A .

As a result of this process, one can verify that the kernel K_3 of the boundary map $R_3 \rightarrow R_2$ vanishes in degrees less than 12. So we can write $K_3 = \Sigma^{12,0} D$, where D is concentrated in non-negative degrees. Note that D is \mathbb{M}_2 -free since it is a submodule of the \mathbb{M}_2 -free module $\Sigma^{-12,0} R_3$.

Proposition 5.5 implies that each R_n is $A(0)$ -free. Then Corollary 3.3 implies that D is $A(0)$ -free as well.

We already know from an earlier step that $\text{Ext}_A^{s,f,w}(D, \mathbf{C}_\eta)$ vanishes for $s < 2f$ and $f \leq 4$. We have isomorphisms

$$\text{Ext}_A^{s,f,w}(D, \mathbf{C}_\eta) \cong \text{Ext}_A^{s+12,f,w}(K_3, \mathbf{C}_\eta) \cong \text{Ext}_A^{s+8,f+4,w}(A(0), \mathbf{C}_\eta)$$

for $f > 0$, so it follows that $\text{Ext}_A^{s,f,w}(A(0), \mathbf{C}_\eta)$ vanishes for $s < 2f$ and $f \leq 8$.

As before, we can then show that when M is an arbitrary module, $\text{Ext}_A^{s,f,w}(M, \mathbf{C}_\eta)$ vanishes for $s < 2f$ and $f \leq 8$. This establishes the proposition for $f \leq 8$.

We can repeat this process to establish the proposition for all f . ■

REFERENCES

- [A1] J. F. Adams, *A finiteness theorem in homological algebra*, Proc. Cambridge Philos. Soc. **57** (1961) 31–36.
- [A2] J. F. Adams, *A periodicity theorem in homological algebra*, Proc. Cambridge Philos. Soc. **62** (1966) 365–377.
- [AM1] J. F. Adams and H. R. Margolis, *Modules over the Steenrod algebra*, Topology **10** (1971) 271–282.
- [AM2] M. Andrews and H. Miller, *Inverting the Hopf map in the Adams-Novikov spectral sequence*, preprint, 2014.
- [B] S. Borghesi, *Cohomology operations and algebraic geometry*, Proceedings of the Nishida Fest (Kinosaki 2003), 75–115, Geom. Topol. Monogr. **10**, 2007.
- [DI] D. Dugger and D. C. Isaksen, *The motivic Adams spectral sequence*, Geom. Topol. **14** (2010) 967–1014.
- [DM] D. Davis and M. Mahowald, *v_1 - and v_2 -periodicity in stable homotopy theory*, Amer. Jour. Math. **104** (1981), no. 3, 615–659.
- [GI] B. Guillou and D. C. Isaksen, *The η -local motivic sphere*, preprint, 2014.
- [I1] D. C. Isaksen, *Stable stems*, preprint, 2014.
- [I2] D. C. Isaksen, *Classical and motivic Adams charts*, preprint, 2014.
- [M] J. Milnor, *The Steenrod algebra and its dual*, Ann. of Math. **67** (1958) 150–171.
- [MM] J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. **81** (1965) 211–264.
- [V1] V. Voevodsky, *Motivic cohomology with $\mathbf{Z}/2$ -coefficients*, Publ. Math. Inst. Hautes Études Sci. No. **98** (2003), 59–104.
- [V2] V. Voevodsky, *Motivic Eilenberg-MacLane spaces*, Publ. Math. Inst. Hautes Études Sci. No. **112** (2010), 1–99.
- [V3] V. Voevodsky, *Reduced power operations in motivic cohomology*, Publ. Math. Inst. Hautes Études Sci. No. **98** (2003), 1–57.

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